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Some elliptic integrals of Barton and Bushell

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Abstract. The main object of the present paper is to derive a unification (and generalization) of certain interesting formulae from Barton (1983 *Proc. R. Soc.* A **388** 445–56) and Bushell (1987 *Math. Proc. Cambridge Phil. Soc.* **101** 1–5) associated with the complete elliptic integrals of the first and second kind. Relevant connections between the results presented here with those given earlier by other workers on the subject are also pointed out. Various families of generalized elliptic integrals, and indeed also definite integrals of such families with respect to their modulus, are known to arise (among other places) in the studies of crystallographic minimal surfaces (cf, e.g., Cvijović and Klinowski (1994 *Proc. R. Soc.* A **444** 525–32)) and in the theory of scattering of accoustic or electromagnetic waves by means of an elliptic disk (cf Björkberg and Kristensson (1987 *Can. J. Phys.* **65** 723–34)).

1. Introduction

In the usual notation, let $K(k)$ and $E(k)$ denote, respectively, the complete elliptic integrals of the first and second kind with *modulus* k (cf, e.g., Byrd and Friedman 1971). Also, for convenience, let $\kappa := \sqrt{1-k^2}$ denote the *complementary modulus* instead of k'.

By evaluating the first term in a certain Born series in two different ways and comparing the resulting expressions, Barton (1983) found the integral formula:

$$\int_0^1 k^\mu K(\kappa)\,\mathrm{d}k = \frac{\pi}{4}\left\{\frac{\Gamma(\frac{1}{2}\mu+\frac{1}{2})}{\Gamma(\frac{1}{2}\mu+1)}\right\}^2 \qquad (\mu > -1). \tag{1.1}$$

Subsequently, while addressing Barton's problem of finding a *direct* proof of his formula (1.1), Bushell (1987) not only proved Barton's formula (1.1) *directly*, but also derived a number of additional results analogous to (1.1), thereby extending several known integral formulae recorded, for example, by Byrd and Friedman (1971) (p 274) (see also Müller 1926, Kaplan 1950). Furthermore, Bushell (1987) (p 2, equation (2.2)) gave a generalization of Barton's integral formula (1.1) in the form:

$$\int_0^1 k^\mu H(\kappa,\gamma)\,\mathrm{d}k = \frac{\pi}{4}\frac{\Gamma(\frac{1}{2}\mu+\frac{1}{2})\Gamma(\gamma+\frac{1}{2}\mu+\frac{1}{2})}{\Gamma(\frac{1}{2}\mu+1)\Gamma(\gamma+\frac{1}{2}\mu+1)} \qquad (\mu > -1) \tag{1.2}$$

where

$$H(k,\gamma) := \int_0^1 (1-t^2)^{-\frac{1}{2}}(1-k^2t^2)^{\gamma-\frac{1}{2}}\,\mathrm{d}t \qquad (\gamma \geqslant 0) \tag{1.3}$$

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so that, obviously,

$$H(k,0) = K(k) \qquad \text{and} \qquad H(k,1) = E(k). \tag{1.4}$$

Making use of (1.2), Bushell (1987) (section 4) proved a general theorem which he applied to deduce numerous further results including, for example, an interesting integral formula associated with the generalized hypergeometric function ${}_pF_q$ (cf, e.g., Erdélyi *et al* (1953, ch 4); see also Srivastava and Karlsson (1985) (p 19 *et seq*). We choose to recall this general integral formula of Bushell (1987) (p 5, corollary) in the following corrected form:

$$\int_0^1 {}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{matrix}k^2\right] H(\kappa,\gamma)k^\nu\,\mathrm{d}k = \frac{\pi}{4}\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2})\Gamma(\gamma+\frac{1}{2}\nu+\frac{1}{2})}{\Gamma(\frac{1}{2}\nu+1)\Gamma(\gamma+\frac{1}{2}\nu+1)}$$
$$\times\ {}_{p+2}F_{q+2}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p, & \frac{1}{2}\nu+\frac{1}{2}, & \gamma+\frac{1}{2}\nu+\frac{1}{2};\\ \beta_1,\ldots,\beta_q, & \frac{1}{2}\nu+1, & \gamma+\frac{1}{2}\nu+1;\end{matrix}1\right] \tag{1.5}$$

which holds true for suitably restricted values of the various parameters involved.

Yet another integral formula analogous to (1.2), also proved by Bushell (1987), may be recalled here as follows (cf Bushell (1987) p 3, equation (2.5)):

$$\int_0^1 k^\mu H(k,\gamma)\,\mathrm{d}k = \frac{\pi}{2(\mu+1)}\,{}_3F_2\left[\begin{matrix}\frac{1}{2}-\gamma, & \frac{1}{2}\mu+\frac{1}{2}, & \frac{1}{2};\\ & 1, & \frac{1}{2}\mu+\frac{3}{2};\end{matrix}1\right] \tag{1.6}$$

or, equivalently,

$$\int_0^1 k^\mu H(k,\gamma)\,\mathrm{d}k = \frac{\pi}{4}\frac{\Gamma(\frac{1}{2}\mu+\frac{1}{2})\Gamma(\gamma+1)}{\Gamma(\frac{1}{2}\mu+1)\Gamma(\gamma+\frac{3}{2})}\,{}_3F_2\left[\begin{matrix}\frac{1}{2}, & \gamma+\frac{1}{2}, & \frac{1}{2}-\frac{1}{2}\mu;\\ & \gamma+\frac{3}{2}, & 1;\end{matrix}1\right] \tag{1.7}$$

and (cf Bushell (1987, p 3, equation (2.6)))

$$\int_0^1 k^\mu H(k,\gamma)\,\mathrm{d}k = \frac{\sqrt{\pi}\Gamma(\gamma+1)}{2\Gamma(\gamma+\frac{3}{2})}\,{}_3F_2\left[\begin{matrix}\gamma+1, & \frac{1}{2}-\frac{1}{2}\mu, & 1;\\ & \gamma+\frac{3}{2}, & \frac{3}{2};\end{matrix}1\right] \tag{1.8}$$

it being understood, in each case, that $\mu > -1$ (and, by definition, $\gamma \geqslant 0$).

The integral formulae (1.7) and (1.8) would result from (1.6) upon iteratively applying the familiar transformation (cf, e.g., Hardy (1923) (p 499, equation (6.1)):

$${}_3F_2\left[\begin{matrix}\alpha,\beta,\gamma;\\ \delta,\epsilon;\end{matrix}1\right] = \frac{\Gamma(\epsilon)\Gamma(\delta+\epsilon-\alpha-\beta-\gamma)}{\Gamma(\epsilon-\gamma)\Gamma(\delta+\epsilon-\alpha-\beta)}\,{}_3F_2\left[\begin{matrix}\delta-\alpha, & \delta-\beta, & \gamma;\\ \delta, & \delta+\epsilon-\alpha-\beta;\end{matrix}1\right]$$
$$(\mathrm{Re}(\delta+\epsilon-\alpha-\beta-\gamma)>0;\ \mathrm{Re}(\epsilon-\gamma)>0). \tag{1.9}$$

Motivated by the usefulness of each of the above integral formulae, as demonstrated by (among others) Barton (1983) and Bushell (1987), we aim here at proving a general theorem which unifies as well as extends *all* of these integral formulae. We also indicate the relevant connections between the results presented here with those given earlier by other workers on the subject.

It should be remarked in passing that an excellent source of simpler cases of integral formulae of the types considered here happens to be the recent work by Prudnikov *et al* (1990) in which most (if not all) of the classical results on the subject can be found. More importantly, various interesting families of generalized elliptic integrals, and indeed also definite integrals of such families with respect to their modulus, are known to arise naturally in a number of seemingly diverse physical contexts; for instance, in the studies of crystallographic minimal surfaces (cf, e.g., Cvijović and Klinowski 1994) and in the theory of scattering of accoustic or electromagnetic waves by means of an elliptic disk (cf Björkberg and Kristensson 1987).

2. A general theorem

First of all, we note that (since $|k| < 1$) definition (1.3) would remain valid even when the parameter γ is unrestricted, in general. Thus, when $\gamma \in \mathbb{R}$ (or, more generally, when $\gamma \in \mathbb{C}$), it is easily seen from (1.3) that

$$H(k, -1) = \frac{E(k)}{\kappa^2} \qquad \left(\kappa := \sqrt{1-k^2}\right) \tag{2.1}$$

since (Bushell 1987, p 2, equation (2.4))

$$H(k, \gamma) = \tfrac{1}{2}\pi\, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2} - \gamma; 1; k^2) \qquad (|k| < 1) \tag{2.2}$$

or, equivalently,

$$H(k, \gamma) = \tfrac{1}{2}\pi (1-k^2)^{\gamma}\, {}_2F_1(\tfrac{1}{2}, \gamma + \tfrac{1}{2}; 1; k^2) \qquad (|k| < 1) \tag{2.3}$$

where we have used Euler's transformation (Erdélyi *et al* 1953, p 64):

$$\begin{aligned} {}_2F_1(a, b; c; z) &= (1-z)^{c-a-b}\, {}_2F_1(c-a, c-b; c; z) \\ &\quad (|\arg(1-z)| < \pi;\ c \neq 0, -1, -2, \ldots). \end{aligned} \tag{2.4}$$

In fact, if we make use of the explicit representation (2.3), we can easily obtain the following (hitherto unnoticed) equivalent form of the integral formulae (1.6), (1.7) and (1.8):

$$\begin{aligned} \int_0^1 k^{\mu} H(k, \gamma)\, \mathrm{d}k &= \frac{\pi}{4} \frac{\Gamma(\frac{1}{2}\mu + \frac{1}{2})\Gamma(\gamma+1)}{\Gamma(\gamma + \frac{1}{2}\mu + \frac{3}{2})}\, {}_3F_2\left[\begin{matrix} \gamma + \frac{1}{2}, \ \frac{1}{2}\mu + \frac{1}{2}, \ \frac{1}{2}; \\ \gamma + \frac{1}{2}\mu + \frac{3}{2}, \ 1; \end{matrix}\ 1\right] \\ &\quad (\mu > -1;\ \gamma > -1) \end{aligned} \tag{2.5}$$

which would also result from (for example) (1.6) when we apply transformation (1.9) with an obviously different choice for the parameters α, β, γ, δ and ϵ involved in (1.6). Furthermore, Bushell's generalization (1.2) of several results including Barton's integral (1.1) can be proven directly by using the explicit representation (2.2) or (2.3) in conjunction with Gauss's summation theorem (Erdélyi *et al* 1953, p 104):

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad (\mathrm{Re}(c-a-b) > 0;\ c \neq 0, -1, -2, \ldots). \tag{2.6}$$

Our unification (and generalization) of each of the aforementioned integral formulae (considered, for example, by Barton (1983), Bushell (1987), and others) is contained in the following theorem.

Theorem. If

$$\Phi(z) := \sum_{n=0}^{\infty} a_n z^n \qquad (|z| < 1) \tag{2.7}$$

and

$$\sum_{n=1}^{\infty} \left| \frac{a_n}{n^{1+\frac{1}{2}\sigma}} \right| < \infty \qquad (\mathrm{Re}(\sigma) > -2) \tag{2.8}$$

then

$$\int_0^1 k^\rho \kappa^\sigma \Phi(zk) H(\zeta\kappa, \gamma)\,\mathrm{d}k = \frac{\pi}{4}\Gamma(\tfrac{1}{2}\sigma + 1)\sum_{n=0}^{\infty} \frac{\Gamma[\frac{1}{2}(\rho+n+1)]}{\Gamma[\frac{1}{2}(\rho+\sigma+n+3)]} a_n z^n$$
$$\times\, {}_3F_2\left[\begin{matrix} \frac{1}{2}-\gamma, & \frac{1}{2}\sigma+1, & \frac{1}{2}; \\ \frac{1}{2}(\rho+\sigma+n+3), & 1; & \end{matrix}\, \zeta^2\right] \qquad \left(\kappa := \sqrt{1-k^2}\right) \tag{2.9}$$

provided further that $\mathrm{Re}(\rho) > -1$ and

$$|\zeta| < 1 \text{ (or } |\zeta| = 1 \text{ and } \mathrm{Re}(\rho + 2\gamma) > -1).$$

Proof. Our proof of assertion (2.9) is based upon the following consequence of the explicit representation (2.2):

$$\int_0^1 k^\rho \kappa^\sigma H(\zeta\kappa, \gamma)\,\mathrm{d}k = \frac{\pi}{4}\frac{\Gamma(\frac{1}{2}\rho+\frac{1}{2})\Gamma(\frac{1}{2}\sigma+1)}{\Gamma[\frac{1}{2}(\rho+\sigma+3)]}\, {}_3F_2\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}-\gamma, & \frac{1}{2}\sigma+1; \\ & \frac{1}{2}(\rho+\sigma+3), & 1; \end{matrix}\, \zeta^2\right]$$
$$\left(\kappa := \sqrt{1-k^2}\right)$$
$$(\mathrm{Re}(\rho) > -1;\ \mathrm{Re}(\sigma) > -2;\ |\zeta| < 1 \text{ (or } |\zeta| = 1 \text{ and } \mathrm{Re}(\rho+2\gamma) > -1)) \tag{2.10}$$

which, for $\rho = \mu$, $\sigma = 0$, and $\zeta = 1$, would reduce to Bushell's formula (1.2) in view of the Gauss summation theorem (2.6). As a matter of fact, by applying a known analytic continuation formula for the Gauss hypergeometric function (Erdélyi *et al* 1953, p 108, equation 2.10(1)), it is easily seen from (2.10) (with $\sigma = 0$) that

$$\int_0^1 k^\rho H(\zeta\kappa, \gamma)\,\mathrm{d}k = \frac{\pi}{4}\frac{\Gamma(\frac{1}{2}\rho+\frac{1}{2})\Gamma(\gamma+\frac{1}{2}\rho+\frac{1}{2})}{\Gamma(\frac{1}{2}\rho+1)\Gamma(\gamma+\frac{1}{2}\rho+1)}\, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}-\gamma; \tfrac{1}{2}-\gamma-\tfrac{1}{2}\rho; 1-\zeta^2)$$
$$+ \frac{\sqrt{\pi}}{4}\frac{\Gamma(\frac{1}{2}\rho+\frac{1}{2})\Gamma(-\gamma-\frac{1}{2}\rho-\frac{1}{2})}{\Gamma(\frac{1}{2}-\gamma)}(1-\zeta^2)^{\gamma+\frac{1}{2}(\rho+1)}$$
$$\times\, {}_2F_1(\tfrac{1}{2}\rho+1, \gamma+\tfrac{1}{2}\rho+1; \gamma+\tfrac{1}{2}\rho+\tfrac{3}{2}; 1-\zeta^2)$$
$$(\mathrm{Re}(\rho) > -1;\ |\arg(1-\zeta^2)| < \pi;\ \gamma+\tfrac{1}{2}\rho+\tfrac{1}{2} \neq 0, \pm 1, \pm 2, \ldots) \tag{2.11}$$

which, for $\rho = \mu$ and $\zeta^2 \to 1$, reduces immediately to Barton's formula (1.2), it being understood that

$$\mu > -1 \qquad (\text{and } \gamma \geqslant 0).$$

If we substitute for $\Phi(zk)$ from (2.7) into the integrand in (2.9), and use term-by-term integration by means of the integral formula (2.10), we shall be led formally to the right-hand side of the assertion (2.9). The formal term-by-term integration can be justified by the *theorem on dominated convergence* under the various conditions stated above, since

$$u_n \sim a_n / n^{1+s+\frac{1}{2}\sigma} \qquad (n \to \infty;\ s \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}) \tag{2.12}$$

where u_n denotes the coefficient of z^n in the series on the right-hand side of (2.9). This evidently completes the proof of the theorem. □

Remark 1. In the special case when $\sigma = 0$, the Clausenian hypergeometric function ${}_3F_2$ occurring on the right-hand side of (2.9) would reduce at once to the Gauss hypergeometric function ${}_2F_1$ which, for $\zeta = 1$, can be summed by means of (2.6), and we thus obtain

$$\int_0^1 k^\rho \Phi(zk) H(\kappa, \gamma)\,\mathrm{d}k = \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{\Gamma[\frac{1}{2}(\rho+n+1)]\Gamma[\gamma+\frac{1}{2}(\rho+n+1)]}{\Gamma[\frac{1}{2}(\rho+n)+1]\Gamma[\gamma+\frac{1}{2}(\rho+n)+1]} a_n z^n \tag{2.13}$$

which holds true under the conditions that are derivable easily from those of the *parent* formula (2.9). A *further* special case of (2.13) when $\rho = 0$ and $z = 1$ happens to be the main theorem in Bushell's paper (1987, p 5, equation (4.1)).

Remark 2. In terms of the generalized hypergeometric function ${}_pF_q$, it is not difficult to deduce from our assertion (2.9) that

$$\begin{aligned}\int_0^1 k^{2\rho}\kappa^{2\sigma}\, {}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p; \\ \beta_1,\ldots,\beta_q;\end{matrix}\, zk^2\right] H(\zeta\kappa,\gamma)\,\mathrm{d}k \\ = \frac{\pi}{4}\frac{\Gamma(\sigma+1)\Gamma(\beta_1)\cdots\Gamma(\beta_q)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_p)} \sum_{n=0}^{\infty} \frac{\Gamma(\rho+n+\frac{1}{2})\Gamma(\alpha_1+n)\cdots\Gamma(\alpha_p+n)}{\Gamma(\rho+\sigma+n+\frac{3}{2})\Gamma(\beta_1+n)\cdots\Gamma(\beta_q+n)} \\ \times {}_3F_2\left[\begin{matrix}\frac{1}{2}-\gamma,\ \sigma+1,\ \frac{1}{2}; \\ \rho+\sigma+n+\frac{3}{2},\ 1;\end{matrix}\, \zeta^2\right]\frac{z^n}{n!} \qquad \left(\kappa := \sqrt{1-k^2}\right)\end{aligned} \tag{2.14}$$

which is valid for suitably restricted values of the various parameters and variables involved. In view of the Gauss summation theorem (2.6), a further special case of our integral formula (2.14) when

$$\sigma = 0 \qquad z = \zeta = 1 \qquad \text{and} \qquad \rho = \tfrac{1}{2}\nu$$

would lead us to Bushell's result (1.5).

Remark 3. Each of the integral formulae (2.9), (2.10), (2.11) and (2.14) can be rewritten in an alternative form by the following simple change of variables:

$$k \longrightarrow \sqrt{1-k^2} \qquad \text{and} \qquad \mathrm{d}k \longrightarrow -\frac{k}{\kappa}\,\mathrm{d}k.$$

For example, (2.9) thus becomes

$$\begin{aligned}\int_0^1 k^{\sigma+1}\kappa^{\rho-1}\Phi(z\kappa)H(\zeta k,\gamma)\,\mathrm{d}k = \frac{\pi}{4}\Gamma(\tfrac{1}{2}\sigma+1)\sum_{n=0}^{\infty}\frac{\Gamma[\frac{1}{2}(\rho+n+1)]}{\Gamma[\frac{1}{2}(\rho+\sigma+n+3)]}a_n z^n \\ \times {}_3F_2\left[\begin{matrix}\frac{1}{2}-\gamma,\ \frac{1}{2}\sigma+1,\ \frac{1}{2}; \\ \frac{1}{2}(\rho+\sigma+n+3),\ 1;\end{matrix}\,\zeta^2\right] \qquad \left(\kappa := \sqrt{1-k^2}\right)\end{aligned} \tag{2.15}$$

which holds true under the same hypotheses as those of the above theorem. This last formula (2.15) and its consequences, analogously to (2.10), (2.11) and (2.14), would generalize a number of integrals belonging to the family represented by (1.6), (1.7), (1.8) and (2.5). For example, (2.15) with

$$\rho = 1 \qquad \sigma = \mu - 1 \qquad z = 0 \qquad \text{and} \qquad \zeta = 1$$

immediately yields the integral formula (1.6).

3. Connections with other elliptic integrals

Motivated by their importance or potential for applications in radiation physics, several recent works were devoted exclusively to the study of various interesting generalizations of the complete elliptic integrals $K(k)$ and $E(k)$. For example, Epstein and Hubbell (1963) (and, subsequently, Weiss (1964)) studied the following family of elliptic integrals:

$$\Omega_j(k) := \int_0^\pi (1 - k^2 \cos\theta)^{-j-\frac{1}{2}}\, d\theta \qquad (j \in \mathbb{N}_0;\ 0 \leqslant k < 1) \tag{3.1}$$

which were encountered in a Legendre polynomial expansion method when applied to certain problems involving computation of the radiation field off-axis from a uniform circular disk radiating according to an arbitrary distribution law *and* which were further extended by Kalla and Al-Saqabi (1991) to allow the parameter j to take on *complex* values. In fact, we have

$$\Omega_0(k) = \frac{\omega\sqrt{2}}{k} K(\omega) \qquad \text{and} \qquad \Omega_1(k) = \frac{\omega\sqrt{2}}{k(1-k^2)} E(\omega) \qquad \left(\omega^2 := \frac{2k^2}{1+k^2}\right). \tag{3.2}$$

More interestingly, by setting

$$t = \cos(\tfrac{1}{2}\theta) = \sqrt{\frac{1+\cos\theta}{2}}$$

it is easily seen from the definition (1.3) that

$$H(k, \gamma) = \tfrac{1}{2}(1 - \tfrac{1}{2}k^2)^{\gamma-\frac{1}{2}} \int_0^\pi \left(1 - \frac{k^2}{2-k^2}\cos\theta\right)^{\gamma-\frac{1}{2}} d\theta$$

which leads us to the following relationship between $H(k, \gamma)$ and $\Omega_\mu(k)$ ($\mu \in \mathbb{C}$):

$$H(k, \gamma) = \frac{(2-k^2)^{\gamma-\frac{1}{2}}}{2^{\gamma+\frac{1}{2}}} \Omega_{-\gamma}\left(\frac{k}{\sqrt{2-k^2}}\right) \qquad (k^2 < 1;\ \gamma \text{ arbitrary}). \tag{3.3}$$

If, in Bushell's definition (1.3), we set $t = \sin\phi$, we immediately obtain

$$H(k, \gamma) = \int_0^{\pi/2} (1 - k^2 \sin^2\phi)^{\gamma-\frac{1}{2}}\, d\phi \qquad (k^2 < 1;\ \gamma \geqslant 0). \tag{3.4}$$

Comparing (3.4) with Das's definition (1987, p 77, equation (7.1)):

$$H_\nu(k) := \int_0^{\pi/2} (1 - k^2 \sin^2\phi)^\nu\, d\phi \qquad (k^2 < 1;\ \nu \text{ arbitrary}) \tag{3.5}$$

so that, obviously,

$$H_{-\frac{1}{2}}(k) = K(k) \qquad \text{and} \qquad H_{\frac{1}{2}}(k) = E(k) \qquad (k^2 < 1) \tag{3.6}$$

we get the relationship:

$$H(k,\gamma) = H_{\gamma-\frac{1}{2}}(k) \qquad (k^2 < 1;\ \gamma \text{ arbitrary}). \tag{3.7}$$

Making use of the relationships (3.3) and (3.7), each of the integral formulae involving $H(k,\gamma)$ can be rewritten at once as an integral involving $\Omega_{-\gamma}(k/\sqrt{2-k^2})$ or $H_{\gamma-\frac{1}{2}}(k)$. Thus, for example, our assertions (2.9) and (2.15) yield the following alternative forms:

$$\int_0^1 k^\rho \kappa^\sigma (2-\zeta^2\kappa^2)^{-\mu-\frac{1}{2}} \Phi(zk) \Omega_\mu \left(\frac{\zeta\kappa}{\sqrt{2-\zeta^2\kappa^2}}\right) \mathrm{d}k$$
$$= \frac{\pi\Gamma(\frac{1}{2}\sigma+1)}{2^{\mu+\frac{3}{2}}} \sum_{n=0}^{\infty} \frac{\Gamma[\frac{1}{2}(\rho+n+1)]}{\Gamma[\frac{1}{2}(\rho+\sigma+n+3)]} a_n z^n \, {}_3F_2\left[\begin{matrix} \mu+\frac{1}{2},\ \frac{1}{2}\sigma+1,\ \frac{1}{2}; \\ \frac{1}{2}(\rho+\sigma+n+3),\ 1; \end{matrix} \zeta^2\right]$$
$$\left(\kappa := \sqrt{1-k^2}\right) \tag{3.8}$$

and

$$\int_0^1 k^{\sigma+1} \kappa^{\rho-1} (2-\zeta^2 k^2)^{-\mu-\frac{1}{2}} \Phi(z\kappa)\, \Omega_\mu \left(\frac{\zeta k}{\sqrt{2-\zeta^2 k^2}}\right) \mathrm{d}k$$
$$= \frac{\pi\Gamma(\frac{1}{2}\sigma+1)}{2^{\mu+\frac{3}{2}}} \sum_{n=0}^{\infty} \frac{\Gamma[\frac{1}{2}(\rho+n+1)]}{\Gamma[\frac{1}{2}(\rho+\sigma+n+3)]} a_n z^n \, {}_3F_2\left[\begin{matrix} \mu+\frac{1}{2},\ \frac{1}{2}\sigma+1,\ \frac{1}{2}; \\ \frac{1}{2}(\rho+\sigma+n+3),\ 1; \end{matrix} \zeta^2\right]$$
$$\left(\kappa := \sqrt{1-k^2}\right) \tag{3.9}$$

respectively, each of which would hold true under the same hypotheses as those of the above theorem *with*, of course, $\gamma = -\mu$.

A special case of this last integral formula (3.9) when

$$\sigma \longrightarrow \sigma - 1 \qquad \rho = 2\lambda + 1 \qquad z = 0 \qquad \text{and} \qquad \zeta = 1$$

happens to be the *main* integral formula of Kalla and Al-Saqabi (1991, p 511, equation (18)).

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